BeamDyn inputs from sectional beam properties

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1 Coordinate system

The coordinate system assumed for BeamDyn in this document is illustrated in Figure 1. This coordinate system is similar to the one used by OpenFAST, with the exception that in this document angles are assumed positive around z, while in OpenFAST the angles are negative around z. As a result of this, all angles θ used in this document, need to be multiplied by -1 for applications with OpenFAST or BeamDyn. The OpenFAST coordinate system together with other aero-elastic codes coordinate systems are shown in Figure 3. The blade coordinate system includes rigid-body pitch rotation relative to the coned coordinate system. The position/orientation, velocity, and acceleration of the blade coordinate system are passed from ElastoDyn to BeamDyn every time step.

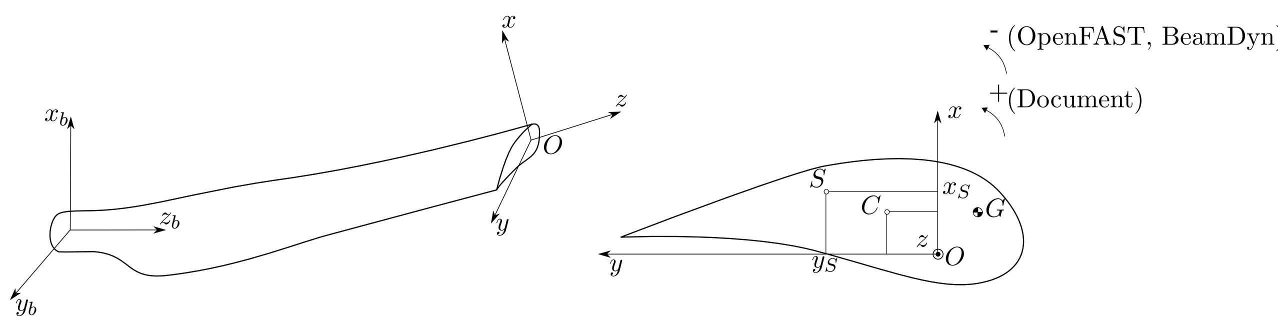

Figure 1: Coordinate system assumed in this document for BeamDyn. Coordinates x_b,y_b,z_b are the coordinates at the root of the blade, used to define the reference axis of the beam. Coordinates x, y, z are used for a typical cross section, with z normal to the cross section, tangent to the reference axis. Point O is the origin of the cross section (defined by the reference axis), point C the centroid, point S the shear center and point G the center of gravity. The coordinates of each points are expressed with respect to the origin of the cross section.

2 Stiffness matrix

The results presented in this section are inspired by the expressions given in the *Dymore* manual [1] but for a different coordinate system[1].

[1]Note that the signs in equation (11) of [1] are erroneous and have to be reversed. Apart from that, the coordinate mapping from this reference to the current document is simply $1 \to z$, $2 \to x$, $3 \to y$.

2.1 Introduction

The stifness matrix that relates the cross-section strains to the loads is asumed to take the following form (valid for orthotropic layups, that is, with no off-axis plies):

$$\begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & K_{16} \\ K_{21} & K_{22} & 0 & 0 & 0 & K_{26} \\ 0 & 0 & K_{33} & K_{34} & K_{35} & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} & 0 \\ 0 & 0 & K_{53} & K_{54} & K_{55} & 0 \\ K_{61} & K_{62} & 0 & 0 & 0 & K_{66} \end{bmatrix} \begin{bmatrix} \gamma_x \\ \gamma_y \\ \epsilon_z \\ \kappa_x \\ \kappa_y \\ \kappa_z \end{bmatrix} \begin{matrix} \text{(flapwise shear)} \\ \text{(edgewise shear)} \\ \text{(axial strain)} \\ \text{(edgewise bending curvature)} \\ \text{(flapwise bending curvature)} \\ \text{(torsion)} \end{matrix} \tag{1}$$

where the text in parenthesis follows the convention used in Figure 1 to draw the airfoil. When the stiffness matrix takes such form, the elements of the matrix involved in the axial and bending loads (F_z, M_x, M_y) and the elements of the matrix involved in the shear-torsion loads (F_x, F_y, M_z) may be determined independently. The two following sections consider these two independent problems. The elements of the matrix for both problems are expressed by considering the centroid and the principal axes of bending for the first problem, and the shear center and the principal shear directions for the second problem.

2.2 Axial force and bending moments

The axial force is decoupled from the bending moments at the centroid (also referred to as the neutral axis or elastic axis) of the cross section. Futher, the bending moments are decoupled when expressed about the principal axes. In this paragraph, the axial forces and bending moments are first expressed with respect to the centroid and the principal axes, then they are rotated to the cross section axis, and last translated to the origin of the cross section. The coordinates of the centroid, expressed from the origin of the cross sections are written (x_C, y_C). The principal axes, noted $\hat{x}_p, \hat{y}_p$, are rotated compared to the cross section axis with an angle θ_p. The axial loads and bending moments about the centroid, and with respect to the principal axes of the cross section are:

$$\begin{bmatrix} F_z^C \\ M_{x_p}^C \\ M_{y_p}^C \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{x_p} & 0 \\ 0 & 0 & EI_{y_p} \end{bmatrix} \begin{bmatrix} \epsilon_z^C \\ \kappa_{x_p}^C \\ \kappa_{y_p}^C \end{bmatrix} \tag{2}$$

where the superscript C indicates that the loads, stress and curvature are expressed at the centroid and the subscript p indicates that the values are expressed in the frame of the princpal axes. The transformation between the principal axes and the cross-section axis is such that:

$$\begin{bmatrix} F_z^C \\ M_x^C \\ M_y^C \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{x_p} & 0 \\ 0 & 0 & EI_{y_p} \end{bmatrix} \boldsymbol{T}^T \begin{bmatrix} \epsilon_z^C \\ \kappa_x^C \\ \kappa_y^C \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & H_{xx} & -H_{xy} \\ 0 & -H_{xy} & H_{yy} \end{bmatrix} \begin{bmatrix} \epsilon_z^C \\ \kappa_x^C \\ \kappa_y^C \end{bmatrix}, \tag{3}$$

with

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_p & -\sin\theta_p \\ 0 & \sin\theta_p & \cos\theta_p \end{bmatrix} \tag{4}$$

and where the variables H_{xx}, H_{xy}, H_{yy} represent the bending stiffnesses in the cross-section coordinates system, obtained from the principal axis stifnesses as:

$$H_{xx} = EI_{x_p} \cos^2 \theta_p + EI_{y_p} \sin^2 \theta_p \tag{5}$$

$$H_{yy} = EI_{x_p} \sin^2 \theta_p + EI_{y_p} \cos^2 \theta_p \tag{6}$$

$$H_{xy} = -EI_{x_p} \sin \theta_p \cos \theta_p + EI_{y_p} \sin \theta_p \cos \theta_p \tag{7}$$

The transformation of the loads, axial stress and curvatures between the origin of the section and the centroid are such that:

$$\begin{bmatrix} F_z \\ M_x \\ M_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ y_C & 1 & 0 \\ -x_C & 0 & 1 \end{bmatrix} \begin{bmatrix} F_z^C \\ M_x^C \\ M_y^C \end{bmatrix}, \qquad \begin{bmatrix} \epsilon_z^C \\ \kappa_x^C \\ \kappa_y^C \end{bmatrix} = \begin{bmatrix} 1 & y_C & -x_C \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_z \\ \kappa_x \\ \kappa_y \end{bmatrix} \tag{8}$$

Using Figure 2, the signs can be verified as follows: a positive axial force at point C ($F_z^C > 0$), leads to a negative moment about y and a positive moment about x at the origin; a positive x-curvature of the beam at the origin ($\kappa_x > 0$), leads to an elongation of the fibers at C, whereas a positive y-curvature of the beam at the origin implies a compression of the fibers at point C ($\epsilon_z^C < 0$) Combining Equation 8 and Equation 3 leads to:

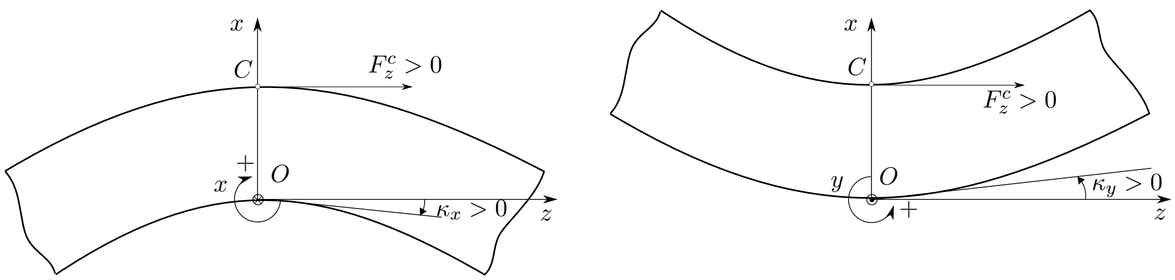

Figure 2: Strain and loads transformation between the centroid C and the origin O.

$$\begin{bmatrix} F_z \\ M_x \\ M_y \end{bmatrix} = \begin{bmatrix} EA & EA\, y_C & -EA\, x_C \\ EA\, y_C & H_{xx} + EA\, y_C^2 & -H_{xy} - EA\, x_C y_C \\ -EA\, x_C & -H_{xy} - EA\, x_C y_C & H_{yy} + EA\, x_C^2 \end{bmatrix} \begin{bmatrix} \epsilon_z \\ \kappa_x \\ \kappa_y \end{bmatrix} \tag{9}$$

2.3 Twisting moment and shear-forces

The torsional moment and the shear forces are decoupled at the shear-center of the cross section, noted S, and of coordinates (x_S, y_S) with respect to the origin of the cross section. Further, the shear forces are independent when expressed with respect to the principal shear directions (usually taken as the principal axes direction), which are the axes, noted $\hat{x}_s, \hat{y}_s$, obtained by rotating the cross section axes $\hat{x}$, $\hat{y}$ by an angle θ_s about the z-axis. The stiffness matrix about the shear center and in the principal shear directions is then:

$$\begin{bmatrix} F_{x_s}^S \\ F_{y_s}^S \\ M_z^S \end{bmatrix} = \begin{bmatrix} k_x GA & 0 & 0 \\ 0 & k_y GA & 0 \\ 0 & 0 & GK_t \end{bmatrix} \begin{bmatrix} \gamma_{x_s}^S \\ \gamma_{y_s}^S \\ \kappa_z^S \end{bmatrix} \tag{10}$$

where the superscript S indicates that the quantities are expressed at the shear center while the subscript s indicates that the values are related to the principal shear directions. The variables k_x and k_y are the dimensionless shear factor related to shear forces in the $\hat{x}_s$ and $\hat{y}_s$ direction respectively. The stiffness matrix at the shear center is transformed to the cross-section frame leading to:

$$\begin{bmatrix} F_x^S \\ F_y^S \\ M_z^S \end{bmatrix} = \begin{bmatrix} K_{xx} & -K_{xy} & 0 \\ -K_{xy} & K_{yy} & 0 \\ 0 & 0 & GK_t \end{bmatrix} \begin{bmatrix} \gamma_x^S \\ \gamma_y^S \\ \kappa_z^S \end{bmatrix} \tag{11}$$

with

$$K_{xx}/GA = k_{x_s} \cos^2\theta_s + k_{y_s} \sin^2\theta_s \tag{12}$$

$$K_{yy}/GA = k_{x_s} \sin^2\theta_s + k_{y_s} \cos^2\theta_s \tag{13}$$

$$K_{xy}/GA = (k_{y_s} - k_{x_s}) \sin\theta_s \cos\theta_s \tag{14}$$

The loads, strains and twisting rate are transferred from the origin to the shear center as follows:

$$\begin{bmatrix} F_x \\ F_y \\ M_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y_S & x_S & 1 \end{bmatrix} \begin{bmatrix} F_x^S \\ F_y^S \\ M_z^S \end{bmatrix}, \qquad \begin{bmatrix} \gamma_x^S \\ \gamma_y^S \\ \kappa_z^S \end{bmatrix} = \begin{bmatrix} 1 & 0 & -y_S \\ 0 & 1 & x_S \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_x \\ \gamma_y \\ \kappa_z \end{bmatrix} \tag{15}$$

Combining Equation 11 with Equation 15 leads to:

$$\begin{bmatrix} F_x \\ F_y \\ M_z \end{bmatrix} = \begin{bmatrix} K_{xx} & -K_{xy} & -K_{xx}y_S - K_{xy}x_S \\ -K_{xy} & K_{yy} & K_{xy}y_S + K_{yy}x_S \\ -K_{xx}y_S - K_{xy}x_S & K_{xy}y_S + K_{yy}x_S & GK_t + K_{xx}y_S^2 + 2K_{xy}x_Sy_S + K_{yy}x_S^2 \end{bmatrix} \begin{bmatrix} \gamma_x \\ \gamma_y \\ \kappa_z \end{bmatrix} \tag{16}$$

2.4 Summary

The results given in Equation 16 and Equation 9 are combined below to form the 6×6 stiffness matrix expressed at the origin of the cross section:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & K_{16} \\ K_{21} & K_{22} & 0 & 0 & 0 & K_{26} \\ 0 & 0 & K_{33} & K_{34} & K_{35} & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} & 0 \\ 0 & 0 & K_{53} & K_{54} & K_{55} & 0 \\ K_{61} & K_{62} & 0 & 0 & 0 & K_{66} \end{bmatrix} \begin{bmatrix} \gamma_x \\ \gamma_y \\ \epsilon_z \\ \kappa_x \\ \kappa_y \\ \kappa_z \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} K_{11} & K_{12} & K_{16} \\ K_{21} & K_{22} & K_{26} \\ K_{61} & K_{62} & K_{66} \end{bmatrix} = \begin{bmatrix} K_{xx} & -K_{xy} & -K_{xx}y_S - K_{xy}x_S \\ -K_{xy} & K_{yy} & K_{xy}y_S + K_{yy}x_S \\ -K_{xx}y_S - K_{xy}x_S & K_{xy}y_S + K_{yy}x_S & GK_t + K_{xx}y_S^2 + 2K_{xy}x_Sy_S + K_{yy}x_S^2 \end{bmatrix}$$

$$\begin{bmatrix} K_{33} & K_{34} & K_{35} \\ K_{43} & K_{44} & K_{45} \\ K_{53} & K_{54} & K_{55} \end{bmatrix} = \begin{bmatrix} EA & EA\,y_C & -EA\,x_C \\ EA\,y_C & H_{xx} + EA\,y_C^2 & -H_{xy} - EA\,x_Cy_C \\ -EA\,x_C & -H_{xy} - EA\,x_Cy_C & H_{yy} + EA\,x_C^2 \end{bmatrix}$$

$$\begin{aligned} H_{xx} &= EI_{x_p}\cos^2\theta_p + EI_{y_p}\sin^2\theta_p \\ H_{yy} &= EI_{x_p}\sin^2\theta_p + EI_{y_p}\cos^2\theta_p \\ H_{xy} &= (EI_{y_p} - EI_{x_p})\sin\theta_p\cos\theta_p \\ K_{xx}/GA &= k_{x_s}\cos^2\theta_s + k_{y_s}\sin^2\theta_s \\ K_{yy}/GA &= k_{x_s}\sin^2\theta_s + k_{y_s}\cos^2\theta_s \\ K_{xy}/GA &= (k_{y_s} - k_{x_s})\sin\theta_s\cos\theta_s \end{aligned}$$

3 Mass matrix

3.1 General form for a rigid body

The mass matrix of a rigid body expressed at its center of mass is

$$\boldsymbol{M}^G = \begin{bmatrix} M\boldsymbol{I}_3 & 0 \\ 0 & \boldsymbol{J}^G \end{bmatrix} \tag{18}$$

The general form of the mass matrix of a rigid body expressed at a given point O is:

$$\boldsymbol{M}^O = \begin{bmatrix} M\boldsymbol{I}_3 & -M\tilde{\boldsymbol{\rho}} \\ M\tilde{\boldsymbol{\rho}} & \boldsymbol{J}^O \end{bmatrix} \tag{19}$$

where $\boldsymbol{\rho} \triangleq \boldsymbol{r}_{OG}$ is the distance from point O to point G and $\boldsymbol{J}^O$ is the inertia tensor of the body at O, related to the inertia tensor at the COG by: $\boldsymbol{J}^O \triangleq -\int \tilde{\boldsymbol{s}}_P \, \tilde{\boldsymbol{s}}_P \, dm = \boldsymbol{J}_G - M\tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\rho}}$, $\boldsymbol{s}_P = \boldsymbol{r}_{OP}$ is a point of the body and where the tilde notation refer to the *skew symmetric matrix*. Given two vectors $\boldsymbol{u}$ and $\boldsymbol{t}$, the skew symmetric matrix is such that $\tilde{\boldsymbol{u}}\,\boldsymbol{t} = \boldsymbol{u} \times \boldsymbol{t}$. which is written in matricial form as follows:

$$\tilde{\boldsymbol{u}} \triangleq \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \tag{20}$$

3.2 Mass matrix of a cross section

Mass matrix about center of mass G, and about the principal inertia directions:

$$\boldsymbol{M}_i^G = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{x_i} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{y_i} & \\ 0 & 0 & 0 & 0 & 0 & I_p \end{bmatrix}, \tag{21}$$

where $I_p = I_{x_i} + I_{y_i}$. Rotated to be expressed about the cross section axis, this becomes:

$$\boldsymbol{M}^G = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xx} & -I_{xy} & 0 \\ 0 & 0 & 0 & -I_{xy} & I_{yy} & \\ 0 & 0 & 0 & 0 & 0 & I_p \end{bmatrix}, \tag{22}$$

with

$$I_{xx} = I_{x_i} \cos^2\theta_i + I_{y_i} \sin^2\theta_i \tag{23}$$

$$I_{yy} = I_{x_i} \sin^2\theta_i + I_{y_i} \cos^2\theta_i \tag{24}$$

$$I_{xy} = (I_{y_i} - I_{x_i}) \sin\theta_i \cos\theta_i \tag{25}$$

Transferred to the origin:

$$\boldsymbol{M}^O = \begin{bmatrix} m & 0 & 0 & 0 & 0 & -my_G \\ 0 & m & 0 & 0 & 0 & mx_G \\ 0 & 0 & m & my_G & -mx_G & 0 \\ 0 & 0 & my_G & I_{xx} + my_G^2 & -I_{xy} - mx_G y_G & 0 \\ 0 & 0 & -mx_G & -I_{xy} - mx_G y_G & I_{yy} + mx_G^2 & 0 \\ -my_G & mx_G & 0 & 0 & 0 & I_p + mx_G^2 + my_G^2 \end{bmatrix} \tag{26}$$

4 Setting BeamDyn inputs from HAWC2 inputs

To setup a BeamDyn model based on a HAWC2 model, it is convenient to use the mid-chord as a reference line of the beam. This is indeed the reference used in HAWC2, referred to as the "c2_def"-coordinate system. The structural file of HAWC2 contains the location of the center of gravity, shear center, and centroid. The centroid is yet referred to as "elastic-center" (defined as the point where the axial force is decoupled from the bending around x and y). The bending stiffness properties of HAWC2 are defined with respect to the principal axes, rotated by an structural pitch angle θ_s around z, compared to the axis of the cross section. The principal axes and principal shear axis are assumed to be the same. The correspondence between the notations of the current document and HAWC2 is given in Table 1.

Table 1: Correspondence between the current definitions and the inputs from HAWC2 for the mean line, section coordinates and stiffness properties

Current		**HAWC2**
(`kp_xr`) x_O	$=$	y_{c2} (`y-pos`)
(`kp_yr`) y_O	$=$	$-x_{c2}$ ("-"`x-pos`)
(`kp_zr`) z_O	$=$	z_{c2} (`z-pos`)
(`initial_twist`) θ_z	$=$	θ_z(`theta_z`)
x_G	$=$	y_m
y_G	$=$	$-x_m$
x_S	$=$	y_s
y_S	$=$	$-x_s$
x_C	$=$	y_e
y_C	$=$	$-x_e$
EA	$=$	$E\,A$
GK_t	$=$	$G\,K$
k_{x_s}	$=$	k_y
k_{y_s}	$=$	k_x
EI_{x_p}	$=$	EI_y
EI_{y_p}	$=$	EI_x
θ_s	$=$	θ_s
θ_p	$=$	θ_s
θ_i	$=$	θ_p
m	$=$	m
I_{x_i}	$=$	$r_{iy}^2 m$
I_{y_i}	$=$	$r_{ix}^2 m$
I_p	$=$	Km/A

A Typical coordinate systems for different aero-elastic codes

The OpenFAST coordinate system together with other aero-elastic codes coordinate systems are shown in Figure 3. The OpenFAST coordinate system follows the IEC wind turbine convention. Angles are negative about z.

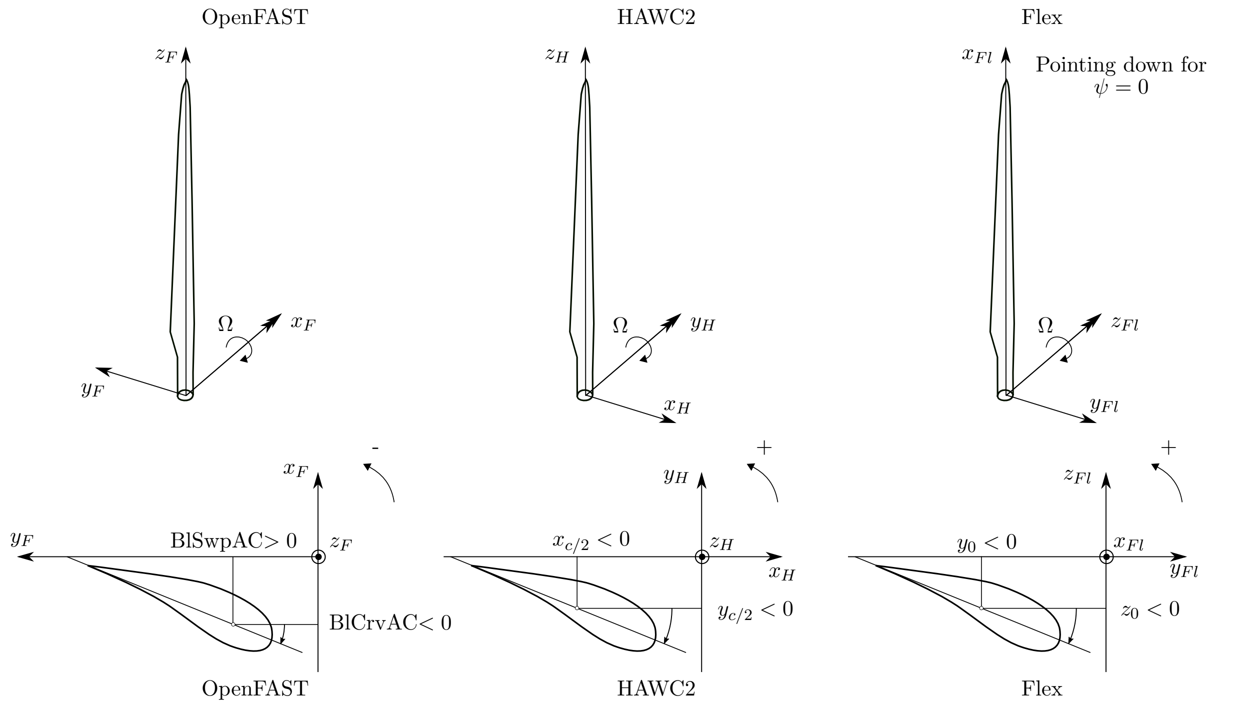

Figure 3: Coordinate systems for three aeroelastic codes: OpenFAST, HAWC2 and Flex. The airfoil cross section is drawn for a typical bending and sweep of an upwind turbine.

References

[1] Bauchau. Dymore manual.